

# Nonconvex minimization related to quadratic double-well energy – approximation by convex problems

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**Abstract.** A double-well energy expressed as a minimum of two quadratic functions, called phase energies, is studied with taking into account the minimization of the corresponding integral functional. Such integral, as being not sequentially weakly lower semicontinuous, does not admit classical minimizers. To derive the relaxation formula for the infimum, the minimizing sequence consisting of solutions of convex problems appropriately approximating the original nonconvex one is constructed. The weak limit of this sequence together with the weak limit of the sequence of solutions of the corresponding dual problems and the weak limits of the characteristic functions related to the phase energies are involved in the relaxation formula.

**Key Words.** Nonconvex minimization problem, Minimum of convex functions, Duality, Parametrized Young measures, Phase transitions.

**AMS Classification.** 49J40, 49M20.

# 1 Introduction

The minimization problem of the form

$$(P) \quad \inf_{v \in H_0^1(\Omega; \mathbb{R}^n)} \int_{\Omega} \min \left\{ \frac{1}{2} a |\varepsilon(v) + C|^2, \frac{1}{2} b |\varepsilon(v) + D|^2 \right\} dx$$

is considered, where  $\varepsilon(v)$  is the symmetrized gradient of  $v \in H_0^1(\Omega; \mathbb{R}^n)$  and  $|\cdot|^2$  stands for the square of the scalar product „ $\cdot$ ” in  $L^2(\Omega; \mathbb{R}_{sym}^{n \times n})$ .

Since the integrand involved is not quasiconvex, we cannot expect the existence of classical minimizers of this functional.

Basically, there are two ways to proceed if there are no minimizers as a consequence of the lack of quasiconvexity ([Mor66]). The first one is to “quasiconvexify” the original functional and relate the gathered information with the functional itself (cf. [Mor66], [Bal77], [BM84], [AF84], [Dac89], [Koh91], [KS86], [Tar75], [Mur79], [Tar79], [Fon88], [FM93], [FR92], [But89], [Dal93], [Amb90], [BBB95], [AF98], [AL99] and the references quoted there). Another possibility is to enlarge the space of admissible functions from Sobolev spaces to the space of parameterized measures (called Young measures [You37]) instead of replacing the objective by its suitable envelopes. In this approach the Young measures can be regarded as means of summarizing the spatial oscillatory properties of minimizing sequences, thus conserving some of that information. With this respect we refer the reader to [You69], [KP91], [CK88], [BJ87], [JK89], [BM84], [Mur79], [Eri80], [Ped97], [Tar91] and the references therein.

However, some important information is lost when seeking minima of lower semicontinuous regularizations (quasiconvex envelopes). Minimizers themselves are not sufficient to characterize properly oscillatory phenomena of the problem (microstructural features describing fine mixtures of the phases in the phase transition problems, for instance). From the application point of view, the detailed structure of minimizing sequences appears to be as important as the minimizers themselves. Moreover, in the vectorial case it is almost impossible to compute for a given objective its quasiconvexification. It is also very difficult to compute explicitly the parametrized measures associated to a minimizing

sequence characterizing the infimum of the problem under consideration.

In this approach a new method to derive the formula for the infimum of  $(P)$  is presented. It preserves all the important information concerning the oscillatory phenomena and is much easier to be obtained in practice. The idea is to approximate  $(P)$  by convex problems as proposed in [Nan01].

**Theorem 1** ([Nan01]). *Suppose that  $f_i : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , are quasiconvex, satisfy the Caratheodory and growth conditions*

(1)  $\forall (s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{nm}$ ,  $\Omega \ni x \mapsto f_i(x, s, \xi)$  is measurable,

(2) for a.e.  $x \in \Omega$ ,  $\mathbb{R}^m \times \mathbb{R}^{nm} \ni (s, \xi) \mapsto f_i(x, s, \xi)$  is continuous,

(3)  $a(x) + c(|s|^2 + |\xi|^2) \leq f_i(x, s, \xi) \leq A(x) + C(|s|^2 + |\xi|^2)$ ,

where  $a(\cdot)$ ,  $A(\cdot)$  are non-negative summable functions in  $\Omega$ ,  $c$  and  $C$  are positive constants.

Set

$$(Q) \quad \inf \left\{ \int_{\Omega} \min \{ f_1(x, u(x), \nabla u(x)), f_2(x, u(x), \nabla u(x)) \} dx : u \in H_0^1(\Omega; \mathbb{R}^n) \right\} := \alpha.$$

Then there exist sequences  $u_k \in H_0^1(\Omega; \mathbb{R}^m)$ ,  $\chi_1^k : \Omega \rightarrow \{0, 1\}$  and  $\chi_2^k : \Omega \rightarrow \{0, 1\}$  with  $\chi_1^k + \chi_2^k \equiv 1$ , such that

(i)  $\{u_k\}$  is a minimizing sequence for  $(Q)$ ,

(ii)  $u_k \rightarrow u$  weakly in  $H_0^1(\Omega; \mathbb{R}^m)$  as  $k \rightarrow \infty$ ,

(iii)  $\chi_1^k \rightarrow \chi_1$ ,  $\chi_2^k \rightarrow \chi_2$  weak\* in  $L^\infty(\Omega)$  as  $k \rightarrow \infty$ ,  $\chi_1 : \Omega \rightarrow [0, 1]$ ,  $\chi_2 : \Omega \rightarrow [0, 1]$  with

$$\chi_1 + \chi_2 \equiv 1,$$

(iv)  $\lim_{k \rightarrow \infty} \int_{\Omega} [\chi_1^k f_1(u_k) + \chi_2^k f_2(u_k)] d\Omega = \alpha$ ,

(v)  $\int_{\Omega} [\chi_1^k f_1(u_k) + \chi_2^k f_2(u_k)] d\Omega \leq \int_{\Omega} [\chi_1^k f_1(w) + \chi_2^k f_2(w)] d\Omega, \quad \forall w \in H_0^1(\Omega; \mathbb{R}^m).$

The minimizing sequence for the problem  $(P)$ , established according to Theorem 1, together with the sequence of solutions of the corresponding dual problems and the sequence of characteristic functions related to the phases  $\frac{1}{2}a|\varepsilon(\cdot) + C|^2$  and  $\frac{1}{2}b|\varepsilon(\cdot) + D|^2$  generate

limits (in appropriate weak topologies) which are involved in the infimum formula. It can be shown that the very special structure of the minimizing sequence to be constructed allows the infimum to be fully expressed by the parametrized Young measures associated to this sequence. Some relations between the related parametrized Young measures and the weak limits of the characteristic functions are established.

## 2 Statement of the problem and its approximation

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain in  $\mathbb{R}^n$  with sufficiently smooth boundary  $\partial\Omega$ . Set

$$\mathcal{J}(u) = \int_{\Omega} \min\left\{\frac{1}{2}a|\varepsilon(u) + C|^2, \frac{1}{2}b|\varepsilon(u) + D|^2\right\} dx, \quad u \in H_0^1(\Omega; \mathbb{R}^n).$$

The problem to be considered here is

$$(P) \quad \inf\left\{\mathcal{J}(u) : u \in H_0^1(\Omega; \mathbb{R}^n)\right\} := \alpha,$$

where  $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a competing vector-valued function from the Sobolev space  $H_0^1(\Omega; \mathbb{R}^n)$ ,  $\varepsilon(u) \in L^2(\Omega; \mathbb{R}_{sym}^{n \times n})$  is the symmetrized gradient of  $u \in H_0^1(\Omega; \mathbb{R}^n)$ ,  $C, D \in L^\infty(\Omega; \mathbb{R}_{sym}^{n \times n})$  and where  $a, b \in L^\infty(\Omega)$  are such that  $a(x), b(x) \geq \delta > 0$  a.e. in  $\Omega$  for a positive constant  $\delta$ .

Theorem 1 ensures the existence of sequences  $u^k \in H_0^1(\Omega; \mathbb{R}^n)$ ,  $\chi_a^k \subset \{0, 1\}$  and  $\chi_b^k \subset \{0, 1\}$ ,  $\chi_a^k + \chi_b^k \equiv 1$ , with the properties that

- (a)  $\{u_k\}$  is a minimizing sequence for (P),
- (b)  $u^k \rightarrow u$  weakly in  $H_0^1(\Omega; \mathbb{R}^n)$  as  $k \rightarrow \infty$ ,
- (c)  $\chi_a^k \rightarrow \chi_a$ ,  $\chi_b^k \rightarrow \chi_b$  weak\* in  $L^\infty(\Omega)$  as  $k \rightarrow \infty$ , where  $\chi_a : \Omega \rightarrow [0, 1]$ ,  $\chi_b : \Omega \rightarrow [0, 1]$  with  $\chi_a + \chi_b \equiv 1$ ,
- (d)  $\int_{\Omega} [\frac{1}{2}\chi_a^k a |\varepsilon(u^k) + C|^2 + \frac{1}{2}\chi_b^k b |\varepsilon(u^k) + D|^2] dx := \alpha^k \rightarrow \alpha$  as  $k \rightarrow \infty$ ,
- (e)  $\int_{\Omega} [\frac{1}{2}\chi_a^k a |\varepsilon(u^k) + C|^2 + \frac{1}{2}\chi_b^k b |\varepsilon(u^k) + D|^2] dx \leq$

$$\leq \int_{\Omega} [\frac{1}{2} \chi_a^k a |\varepsilon(w) + C|^2 + \frac{1}{2} \chi_b^k b |\varepsilon(w) + D|^2] dx, \quad \forall w \in H_0^1(\Omega; \mathbb{R}^n).$$

Let us now introduce the function which describes the behaviour of the minimizing sequence  $\{u_k\}$ :

$$\psi^k = \chi_b^k - \chi_a^k \quad (2.1)$$

with the property

$$(\psi^k)^2 = 1. \quad (2.2)$$

It will be convenient to introduce the notations:

$$m^k := \chi_a^k a + \chi_b^k b, \quad \overline{m} := \frac{a+b}{2}, \quad \underline{m} := \frac{b-a}{2}$$

Notice that  $m^k$  has a decomposition

$$m^k = \overline{m} + \psi^k \underline{m}. \quad (2.3)$$

From (e) of Theorem 1 it follows that  $u^k$  is a solution of the convex optimization problem

$$(P^k) \quad \inf \left\{ \mathcal{J}^k(v) : v \in H_0^1(\Omega; \mathbb{R}^n) \right\} := \alpha^k,$$

where

$$\mathcal{J}^k(v) = \int_{\Omega} \left[ \frac{1}{2} \chi_a^k a |\varepsilon(v) + C|^2 + \frac{1}{2} \chi_b^k b |\varepsilon(v) + D|^2 \right] dx, \quad v \in H_0^1(\Omega; \mathbb{R}^n),$$

i.e.

$$\mathcal{J}^k(u^k) = \alpha^k.$$

Easy calculations using the properties of the scalar product show that

$$\begin{aligned} \chi_a^k a C + \chi_b^k b C &= \frac{aC + bD}{2} + \psi^k \frac{bD - aC}{2}, \\ \frac{1}{2} \chi_a^k a |C|^2 + \frac{1}{2} \chi_b^k b |D|^2 &= \frac{1}{2} \left( \frac{a|C|^2 + b|D|^2}{2} + \psi^k \frac{b|D|^2 - a|C|^2}{2} \right) \end{aligned}$$

so that  $\mathcal{J}^k(\cdot)$  admits the representation

$$\left. \begin{aligned} \mathcal{J}^k(v) &= \int_{\Omega} \left[ \frac{1}{2}(\chi_a^k a + \chi_b^k b) |\varepsilon(v)|^2 + (\chi_a^k a C + \chi_b^k b D) \cdot \varepsilon(v) \right. \\ &\quad \left. + \frac{1}{2} \chi_a^k a |C|^2 + \frac{1}{2} \chi_b^k b |D|^2 \right] dx \\ &= \int_{\Omega} \left[ \frac{1}{2} m^k |\varepsilon(v)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(v) + \frac{1}{2} \mathcal{B}^k \right] dx, \end{aligned} \right\} \quad (\text{R})$$

where the following notations have been introduced

$$\begin{aligned} \mathcal{A}^+ &:= \frac{aC + bD}{2} \\ \mathcal{A}^- &:= \frac{bD - aC}{2} \\ \mathcal{B}^k &:= \frac{a|C|^2 + b|D|^2}{2} + \psi^k \frac{b|D|^2 - a|C|^2}{2}. \end{aligned} \quad (2.4)$$

Now we associate to the minimization problem  $(P^k)$  the dual problem  $(P^k)^*$  following the idea of Fenchel [Fen51] (cf. also [ET76], [Aub93]). Let  $\phi(\cdot)$  be given by

$$\phi(\xi) = \frac{m}{2} |\xi|^2 + E \cdot \xi, \quad \xi \in \mathbb{R}_{sym}^{n \times n}, \quad (2.5)$$

where  $m > 0$  is a constant, then its conjugate  $\phi^c(\cdot)$  reads

$$\phi^c(p) = \sup_{\xi \in \mathbb{R}_{sym}^{n \times n}} (p \cdot \xi - \phi(\xi)) = \frac{1}{2m} |p - E|^2, \quad p \in \mathbb{R}_{sym}^{n \times n}. \quad (2.6)$$

Next define a linear continuous operator  $L : H_0^1(\Omega; \mathbb{R}^n) \rightarrow L^2(\Omega; \mathbb{R}_{sym}^{n \times n})$  as

$$Lv = \varepsilon(v), \quad v \in H_0^1(\Omega; \mathbb{R}^n),$$

then its transpose  $L^* : L^2(\Omega; \mathbb{R}_{sym}^{n \times n}) \rightarrow H^{-1}(\Omega; \mathbb{R}^n)$  can be expressed as

$$\langle L^* p, v \rangle_{H_0^1(\Omega; \mathbb{R}^n)} = \int_{\Omega} p \cdot \varepsilon(v) dx, \quad \forall p \in L^2(\Omega; \mathbb{R}_{sym}^{n \times n}), \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n).$$

Define

$$\mathcal{I}^k(q) := \int_{\Omega} \left[ \frac{1}{2m^k} |q - (\mathcal{A}^+ + \psi^k \mathcal{A}^-)|^2 - \frac{1}{2} \mathcal{B}^k \right] dx, \quad q \in L^2(\Omega; \mathbb{R}_{sym}^{n \times n}) \quad (2.7)$$

and denote by  $\text{Ker } L^*$  the kernel of  $L^*$ , i.e.

$$\text{Ker } L^* = \left\{ p \in L^2(\Omega; \mathbb{R}_{sym}^{n \times n}) : \int_{\Omega} p \cdot \varepsilon(v) dx = 0 \right\}.$$

Now we are in a position to formulate the dual problem  $(P^k)^\star$  which can be stated as follows

$$(P^k)^\star \quad \inf \left\{ \int_{\Omega} \mathcal{I}^k(q) dx : q \in \text{Ker } L^\star \right\} := \beta^k.$$

According to the Fenchel theorem (cf. Theorem 3.2, p. 38, [Aub93]) we get

$$\begin{aligned} \mathcal{J}^k(v) &= \int_{\Omega} \left[ \frac{1}{2} m^k |\varepsilon(v)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(v) + \frac{1}{2} \mathcal{B}^k \right] dx \geq \\ &\geq \int_{\Omega} \left[ \frac{1}{2} m^k |\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) + \frac{1}{2} \mathcal{B}^k \right] dx = \\ &= \mathcal{J}^k(u^k) = \alpha^k = -\beta^k = -\mathcal{I}^k(p^k) = \\ &= - \int_{\Omega} \left[ \frac{1}{2m^k} |p^k - (\mathcal{A}^+ + \psi^k \mathcal{A}^-)|^2 - \frac{1}{2} \mathcal{B}^k \right] dx \geq \\ &\geq - \int_{\Omega} \left[ \frac{1}{2m^k} |q - (\mathcal{A}^+ + \psi^k \mathcal{A}^-)|^2 - \frac{1}{2} \mathcal{B}^k \right] dx = -\mathcal{I}^k(q), \\ &\quad \forall v \in H_0^1(\Omega; \mathbb{R}^n), \forall q \in \text{Ker } L^\star, \end{aligned} \quad (2.8)$$

where

$$p^k = m^k \varepsilon(u^k) + \mathcal{A}^+ + \psi^k \mathcal{A}^- \in \text{Ker } L^\star \quad (2.9)$$

is a solution of the dual problem  $(P^k)^\star$ . Since  $p^k \in \text{Ker } L^\star$ ,

$$\int_{\Omega} p^k \cdot \varepsilon(v) dx = 0, \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n), \quad (2.10)$$

so, in particular,

$$\int_{\Omega} p^k \cdot \varepsilon(u^k) dx = \int_{\Omega} \left[ m^k |\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) \right] dx = 0. \quad (2.11)$$

Taking into account that  $\alpha^k = \mathcal{I}^k(p^k)$  and equation (2.11) we get

the following representations

$$\alpha^k = \frac{1}{2} \int_{\Omega} [-m^k |\varepsilon(u^k)|^2 + \mathcal{B}^k] dx = \frac{1}{2} \int_{\Omega} [(\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) + \mathcal{B}^k] dx. \quad (2.12)$$

Analogously, from (2.9) we have  $\varepsilon(u^k) = \frac{1}{m^k} (p^k - \mathcal{A}^+ - \psi^k \mathcal{A}^-)$ , so from (2.11) and (2.12)

it follows

$$\int_{\Omega} \left[ \frac{1}{m^k} |p^k|^2 - \frac{1}{m^k} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot p^k \right] dx = 0.$$

Therefore

$$\begin{aligned}\alpha^k &= -\frac{1}{2} \int_{\Omega} \left[ -\frac{1}{m^k} |p^k|^2 + \frac{1}{m^k} |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 - \mathcal{B}^k \right] dx = \\ &= \frac{1}{2} \int_{\Omega} \left[ \frac{1}{m^k} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot p^k - \frac{1}{m^k} |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 + \mathcal{B}^k \right] dx\end{aligned}\quad (2.13)$$

and we are led to the equality

$$\int_{\Omega} \left[ m^k |\varepsilon(u^k)|^2 + \frac{1}{m^k} |p^k|^2 \right] dx = \int_{\Omega} \left[ \frac{1}{m^k} |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 \right] dx. \quad (2.14)$$

Now observe that the right hand side of this equality can be reorganized in such a way that its limit as  $k \rightarrow \infty$  is easy to be calculated. Indeed, if we set

$$\overline{m}^{\natural} := \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right), \quad \underline{m}^{\natural} := \frac{1}{2} \left( \frac{1}{b} - \frac{1}{a} \right),$$

and recall that  $\chi_a^k + \chi_b^k = 1$  and  $(\psi^k)^2 = 1$  we obtain

$$m^k = \frac{\overline{m}^{\natural} - \psi^k \underline{m}^{\natural}}{(\overline{m}^{\natural})^2 - (\underline{m}^{\natural})^2}$$

so that

$$\frac{1}{m^k} = \overline{m}^{\natural} + \underline{m}^{\natural} \psi^k$$

and from (2.14), by making use of (2.2), we obtain

$$\begin{aligned}& \lim_{k \rightarrow \infty} \int_{\Omega} \left[ m^k |\varepsilon(u^k)|^2 + \frac{1}{m^k} |p^k|^2 \right] dx \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} \left[ (\overline{m}^{\natural} + \underline{m}^{\natural} \psi^k) |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 \right] dx \\ &= \int_{\Omega} \left[ (\overline{m}^{\natural} + \underline{m}^{\natural} \psi) |\mathcal{A}^+|^2 + 2(\overline{m}^{\natural} \psi + \underline{m}^{\natural}) \mathcal{A}^+ \cdot \mathcal{A}^- + (\overline{m}^{\natural} + \underline{m}^{\natural} \psi) |\mathcal{A}^-|^2 \right] dx \\ &= \frac{1}{2} \int_{\Omega} [(\overline{m}^{\natural} + \underline{m}^{\natural} \psi) (a^2 |C|^2 + b^2 |D|^2) + (\overline{m}^{\natural} \psi + \underline{m}^{\natural}) (b^2 |D|^2 - a^2 |C|^2)] dx \\ &= \frac{1}{2} \int_{\Omega} (a |C|^2 + b |D|^2) dx + \frac{1}{2} \int_{\Omega} \psi (b |D|^2 - a |C|^2) dx = \int_{\Omega} \mathcal{B} dx,\end{aligned}\quad (2.15)$$

where

$$\mathcal{B} = \frac{a |C|^2 + b |D|^2}{2} + \psi \frac{b |D|^2 - a |C|^2}{2}.$$

Let us introduce the set

$$\Omega_0 = \{x \in \Omega : a(x) = b(x)\}.$$



Using (2.9) we obtain immediately that in  $\Omega \setminus \Omega_0$ ,

$$\psi^k \varepsilon(u^k) = \frac{1}{\underline{m}} p^k - \frac{\overline{m}}{\underline{m}} \varepsilon(u^k) - \frac{1}{\underline{m}} \mathcal{A}^+ - \frac{1}{\underline{m}} \psi^k \mathcal{A}^-.$$

Thus from (2.12) it follows

$$\begin{aligned} \alpha^k &= \frac{1}{2} \int_{\Omega \setminus \Omega_0} [\mathcal{A}^+ \cdot \varepsilon(u^k) + \mathcal{A}^- \cdot \psi^k \varepsilon(u^k) + \mathcal{B}^k] dx \\ &\quad + \frac{1}{2} \int_{\Omega_0} [\mathcal{A}^+ \cdot \varepsilon(u^k) + \mathcal{A}^- \cdot \psi^k \varepsilon(u^k) + \mathcal{B}^k] dx \\ &= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ (\mathcal{A}^+ - \frac{\overline{m}}{\underline{m}} \mathcal{A}^-) \cdot \varepsilon(u^k) + \frac{1}{\underline{m}} \mathcal{A}^- \cdot p^k - \frac{1}{\underline{m}} \mathcal{A}^- \cdot \mathcal{A}^+ \right. \\ &\quad \left. - \frac{1}{\underline{m}} \psi^k |\mathcal{A}^-|^2 + \mathcal{B}^k \right] dx \\ &\quad + \frac{1}{2} \int_{\Omega_0} [\mathcal{A}^+ \cdot \varepsilon(u^k) + p^k \cdot \varepsilon(u^k) - a |\varepsilon(u^k)|^2 - \mathcal{A}^+ \cdot \varepsilon(u^k) + \mathcal{B}^k] dx \\ &= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\ &\quad \left. - \psi^k \frac{ab|C-D|^2}{2(b-a)} \right] dx \\ &\quad + \frac{1}{2} \int_{\Omega_0} \left[ -a |\varepsilon(u^k)|^2 + \frac{a(|C|^2 + |D|^2)}{2} + \psi^k \frac{a(|D|^2 - |C|^2)}{2} \right] dx \\ &\quad + \frac{1}{2} \int_{\Omega_0} p^k \cdot \varepsilon(u^k) dx. \end{aligned} \tag{2.16}$$

Another representation of  $\alpha^k$  can be derived from (2.9) if we take into account (2.13),

$(\psi^k)^2 = 1$  and that in  $\Omega_0$ ,  $\frac{1}{a}(|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi^k \mathcal{A}^+ \cdot \mathcal{A}^-) = \mathcal{B}^k$ :

$$\begin{aligned}
\alpha^k &= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ \frac{1}{m^k} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot p^k - \frac{1}{m^k} |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 + \mathcal{B}^k \right] dx \\
&\quad + \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot p^k - \frac{1}{a} |\mathcal{A}^+ + \psi^k \mathcal{A}^-|^2 + \mathcal{B}^k \right] dx \\
&= \frac{1}{2} \int_{\Omega \setminus \Omega_0} [\mathcal{A}^+ \cdot \varepsilon(u^k) + \mathcal{A}^- \cdot \psi^k \varepsilon(u^k) + \mathcal{B}^k] dx \\
&\quad + \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} |p^k|^2 - \frac{1}{a} (|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi^k \mathcal{A}^+ \cdot \mathcal{A}^-) + \mathcal{B}^k \right] dx \\
&= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\
&\quad \left. - \psi^k \frac{ab|C-D|^2}{2(b-a)} \right] dx \\
&\quad + \frac{1}{2} \int_{\Omega_0} \frac{1}{a} |p^k|^2 dx - \frac{1}{2} \int_{\Omega_0} p^k \cdot \varepsilon(u^k) dx. \tag{2.17}
\end{aligned}$$

As it will be seen in the next section the formulas (2.16) and (2.17) allow us to express  $\alpha$  in terms of the weak limits:  $u$ ,  $p$  and  $\psi$ .

### 3 Weak convergence in $L^1(\Omega)$

**Lemma 2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain in  $\mathbb{R}^n$  with Lipschitz continuous boundary  $\partial\Omega$ . Then*

$$p^k \cdot \varepsilon(u^k) \rightarrow p \cdot \varepsilon(u) \quad \text{weakly in } L^1(\Omega). \tag{3.1}$$

*Proof.* Extend each function  $u^k \in H_0^1(\Omega; \mathbb{R}^n)$  to all of  $\mathbb{R}^n$  by setting it equal to zero on  $\mathbb{R}^n \setminus \Omega$ . By regularity of the boundary  $\partial\Omega$  all of these extensions are elements of  $H^1(\mathbb{R}^n; \mathbb{R}^n)$ . For an arbitrary  $\varphi \in C^\infty(\mathbb{R}^n)$  we thus have  $\varphi u^k \in H^1(\mathbb{R}^n; \mathbb{R}^n)$  and  $\varphi u^k|_\Omega \in H_0^1(\Omega; \mathbb{R}^n)$ . We claim that

$$\int_{\mathbb{R}^n} \varphi p^k \cdot \varepsilon(u^k) dx \rightarrow \int_{\mathbb{R}^n} \varphi p \cdot \varepsilon(u) dx, \tag{3.2}$$

for any  $\varphi \in C^\infty(\mathbb{R}^n)$ . Indeed, since  $\varepsilon(\varphi u^k) = \varphi \varepsilon(u^k) + u^k \otimes \nabla \varphi$ , we get

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi p^k \cdot \varepsilon(u^k) dx &= \int_{\mathbb{R}^n} p^k \cdot \varepsilon(\varphi u^k) dx - \int_{\mathbb{R}^n} p^k \cdot (u^k \otimes \nabla \varphi) dx = \\ &= - \int_{\mathbb{R}^n} p^k \cdot (u^k \otimes \nabla \varphi) dx \rightarrow - \int_{\mathbb{R}^n} p \cdot (u \otimes \nabla \varphi) dx = \int_{\mathbb{R}^n} \varphi p \cdot \varepsilon(u) dx, \end{aligned}$$

where we have used (2.10) and the strong convergence  $u^k \rightarrow u$  in  $L^2(\Omega; \mathbb{R}^n)$  (valid due to the Rellich compactness theorem).

Further, the sequence  $\{p^k \cdot \varepsilon(u^k)\}$  is uniformly bounded in  $L^1(\Omega)$  because  $\{p^k\}$  and  $\{\varepsilon(u^k)\}$  so are in  $L^2(\Omega)$ . By Chacon's biting lemma [Ped97] it follows that there exist a subsequence of  $(p^k \cdot \varepsilon(u^k))$ , not relabeled, a nonincreasing sequence of measurable sets  $\Omega_n \subset \Omega$ ,  $\Omega_{n+1} \subset \Omega_n$ ,  $|\Omega_n| \searrow 0$  and  $f \in L^1(\Omega)$  such that

$$p^k \cdot \varepsilon(u^k) \rightarrow f \quad \text{weakly in } L^1(\Omega \setminus \Omega_n) \quad (3.3)$$

for all  $n$ . It means that  $\{p^k \cdot \varepsilon(u^k)\}$  converges in the biting sense to  $f$  [Ped97].

Now we assert that the biting limit  $f$  coincides with  $p \cdot \varepsilon(u)$ , i.e.  $f = p \cdot \varepsilon(u)$  a.e. in  $\Omega$ . To show this observe that from the biting argument (3.3) and (3.2) it follows that for any  $\varphi \in C^\infty(\mathbb{R}^n)$  we get

$$\int_{\Omega \setminus \Omega_n} \varphi p \cdot \varepsilon(u) dx = \int_{\Omega \setminus \Omega_n} \varphi f dx, \quad (3.4)$$

for any  $n$ . Hence  $p \cdot \varepsilon(u) = f$  a.e. in  $\Omega \setminus \Omega_n$  for each  $n$ . Since  $|\Omega_n| \searrow 0$  as  $n \rightarrow \infty$ , the equality  $p \cdot \varepsilon(u) = f$  must hold a.e. in  $\Omega$ . Thus the assertion follows.

Recall that  $p^k \cdot \varepsilon(u^k) = m^k |\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k)$ . Therefore one can deduce the existence of a constant  $C \geq 0$  such that

$$p^k \cdot \varepsilon(u^k) + C \geq 0 \quad \text{a.e in } \Omega.$$

Obviously  $p^k \cdot \varepsilon(u^k) + C$  converges in the biting sense to  $p \cdot \varepsilon(u) + C$ . According to (Lemma 6.9, p.109, [Ped97]) its weak convergence in  $L^1(\Omega)$  is then equivalent to

$$\limsup_{k \rightarrow \infty} \int_{\Omega} (p^k \cdot \varepsilon(u^k) + C) dx \leq \int_{\Omega} (p \cdot \varepsilon(u) + C) dx. \quad (3.5)$$

Our task now is to establish the foregoing inequality. For this purpose notice that (3.2) can be easily extend to the convergence

$$\int_{\mathbb{R}^n} \varphi p^k \cdot \varepsilon(u^k) dx \rightarrow \int_{\mathbb{R}^n} \varphi p \cdot \varepsilon(u) dx, \quad (3.6)$$

which is valid for any  $\varphi \in C_c(\mathbb{R}^n)$ , where  $C_c(\mathbb{R}^n)$  is the space of continuous functions on  $\mathbb{R}^n$  with compact support. Thus  $\mu^k := (p^k \cdot \varepsilon(u^k) + C)dx$  and  $\mu := (p \cdot \varepsilon(u) + C)dx$  can be treated as positive Radon measures on  $\mathbb{R}^n$  for which it holds

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \varphi d\mu^k = \int_{\mathbb{R}^n} \varphi d\mu, \quad \forall \varphi \in C_c(\mathbb{R}^n).$$

But (Theorem 1, p. 54, [EG92]) asserts that this condition is equivalent to the following one

$$\lim_{k \rightarrow \infty} \mu^k(B) = \mu(B) \quad \text{for each bounded Borel set } B \subset \mathbb{R}^n \text{ with } \mu(\partial B) = 0. \quad (3.7)$$

Now we are in a position to show (3.5). Fix  $\epsilon > 0$  and choose  $0 < \delta < \epsilon$  with the property that  $\omega \subset \Omega$  with  $|\omega| < \delta$  implies

$$\int_{\omega} p \cdot \varepsilon(u) dx < \epsilon.$$

In the biting convergence take  $n_0$  large enough to fulfill  $|\Omega_{n_0}| < \frac{\delta}{2}$ . By the measurability of  $\Omega_{n_0}$  there exists an open  $\tilde{\Omega}_{n_0} \supset \Omega_{n_0}$  with  $|\tilde{\Omega}_{n_0}| < \delta$ . Vitali's covering theorem ensures the representation  $\tilde{\Omega}_{n_0} = \tilde{\Omega}'_{n_0} \cup \tilde{\Omega}''_{n_0}$  where  $|\tilde{\Omega}''_{n_0}| = 0$  and  $\tilde{\Omega}'_{n_0}$  stands for the union of a countable collection of disjoint closed balls in  $\tilde{\Omega}_{n_0}$ . Therefore  $|\partial \tilde{\Omega}'_{n_0}| = 0$  and consequently  $\mu(\partial \tilde{\Omega}'_{n_0}) = 0$ . From this we have

$$\begin{aligned} \int_{\Omega} (p^k \cdot \varepsilon(u^k) + C) dx &= \int_{\Omega \setminus \Omega_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx + \int_{\Omega_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx \\ &\leq \int_{\Omega \setminus \Omega_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx + \int_{\tilde{\Omega}_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx \\ &= \int_{\Omega \setminus \Omega_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx + \int_{\tilde{\Omega}'_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx \\ &= \int_{\Omega \setminus \Omega_{n_0}} (p^k \cdot \varepsilon(u^k) + C) dx + \mu^k(\tilde{\Omega}'_{n_0}) \end{aligned}$$

which thanks to (3.7) by passing to the limit as  $k \rightarrow \infty$  yields

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \int_{\Omega} (p^k \cdot \varepsilon(u^k) + C) dx &\leq \int_{\Omega \setminus \Omega_{n_0}} (p \cdot \varepsilon(u) + C) dx + \mu(\tilde{\Omega}'_{n_0}) \\
&\leq \int_{\Omega} (p \cdot \varepsilon(u) + C) dx + \int_{\tilde{\Omega}'_{n_0}} (p \cdot \varepsilon(u) + C) dx \\
&\leq \int_{\Omega} (p \cdot \varepsilon(u) + C) dx + \epsilon(1 + C),
\end{aligned}$$

because  $|\tilde{\Omega}'_{n_0}| < \delta < \epsilon$ . Since  $\epsilon > 0$  was chosen arbitrarily, (3.5) follows. This completes the proof of Lemma 2.  $\square$

## 4 Relaxed formulas for the infimum

The weak lower semicontinuity of convex functionals, the upper semicontinuity of concave functionals and Lemma 2 yield

$$\liminf_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega_0} \frac{1}{a} |p^k|^2 dx \geq \frac{1}{2} \int_{\Omega_0} \frac{1}{a} |p|^2 dx, \quad (4.1)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega_0} [-a |\varepsilon(u^k)|^2 + \mathcal{B}^k] dx \leq \frac{1}{2} \int_{\Omega_0} [-a |\varepsilon(u)|^2 + \mathcal{B}] dx, \quad (4.2)$$

$$\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega_0} p^k \cdot \varepsilon(u^k) dx = \frac{1}{2} \int_{\Omega_0} p \cdot \varepsilon(u) dx, \quad (4.3)$$

where  $\psi = \chi_b - \chi_a$  and

$$\mathcal{B} = \frac{a(|C|^2 + |D|^2)}{2} + \psi \frac{a(|D|^2 - |C|^2)}{2} \quad \text{in } \Omega_0.$$

Now we show that

$$\begin{aligned}
&\lim_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C - D)}{b - a} \cdot \varepsilon(u^k) + \frac{bD - aC}{b - a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b - a)} \right. \\
&\quad \left. - \psi^k \frac{ab|C - D|^2}{2(b - a)} \right] dx \\
&= \frac{1}{2} \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C - D)}{b - a} \cdot \varepsilon(u) + \frac{bD - aC}{b - a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b - a)} \right. \\
&\quad \left. - \psi \frac{ab|C - D|^2}{2(b - a)} \right] dx.
\end{aligned} \quad (4.4)$$

This is not trivial because the functions  $\frac{ab(C-D)}{b-a}$  and  $\frac{bD-aC}{b-a}$  are not assumed to belong to  $L^2(\Omega \setminus \Omega_0; \mathbb{R}_{sym}^{n \times n})$ . To overcome this disadvantage let us recall (see (2.16) or (2.17)) that  $\Omega \setminus \Omega_0$  is a set of a finite Lebesgue measure where

$$\begin{aligned} \frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} - \psi^k \frac{ab|C-D|^2}{2(b-a)} \\ = \mathcal{A}^+ \cdot \varepsilon(u^k) + \mathcal{A}^- \cdot \psi^k \varepsilon(u^k) + \mathcal{B}^k. \end{aligned}$$

Thus for any  $\varepsilon > 0$  there exist  $\omega_\varepsilon \subset \Omega \setminus \Omega_0$  and  $\delta > 0$  such that  $|\omega_\varepsilon| < \varepsilon$  and for each  $x \in (\Omega \setminus \Omega_0) \setminus \omega_\varepsilon$  one has  $|a(x) - b(x)| \geq \delta$ . Hence

$$\frac{C-D}{b-a} \in L^\infty((\Omega \setminus \Omega_0) \setminus \omega_\varepsilon; \mathbb{R}_{sym}^{n \times n}) \subset L^2((\Omega \setminus \Omega_0) \setminus \omega_\varepsilon; \mathbb{R}_{sym}^{n \times n}) \quad (4.5)$$

and

$$\begin{aligned} \left| \int_{\omega_\varepsilon} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \right. \\ \left. \left. - \psi^k \frac{ab|C-D|^2}{2(b-a)} \right] dx \right| \leq \text{const } |\omega_\varepsilon|^{\frac{1}{2}} \leq \text{const } \varepsilon^{\frac{1}{2}}. \end{aligned}$$

This allows the conclusion that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{(\Omega \setminus \Omega_0) \setminus \omega_\varepsilon} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u^k) + \frac{bD-aC}{b-a} \cdot p^k + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\ \left. - \psi^k \frac{ab|C-D|^2}{2(b-a)} \right] dx \\ = \int_{(\Omega \setminus \Omega_0) \setminus \omega_\varepsilon} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u) + \frac{bD-aC}{b-a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\ \left. - \psi \frac{ab|C-D|^2}{2(b-a)} \right] dx \end{aligned}$$

and due to the fact that  $\varepsilon > 0$  was chosen arbitrarily we easily arrive at (4.4), as desired.

Now, for  $v \in H_0^1(\Omega; \mathbb{R}^n)$  and  $q \in \text{Ker } L^*$  let us set

$$\mathcal{I}(v, q) := \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(v) + \frac{bD-aC}{b-a} \cdot q + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} - \psi \frac{ab|C-D|^2}{2(b-a)} \right] dx. \quad (4.6)$$

Using the fact that for all  $k \in \mathbb{N}$  we have  $p^k \in \text{Ker } L^*$ , (4.4) and passing to the limit as  $k \rightarrow \infty$  in (2.16) we get

$$\alpha \leq \frac{1}{2} \mathcal{I}(u, p) + \frac{1}{2} \int_{\Omega_0} \left[ -a |\varepsilon(u)|^2 \right] dx + \frac{1}{2} \int_{\Omega_0} \mathcal{B} dx + \frac{1}{2} \int_{\Omega_0} p \cdot \varepsilon(u) dx.$$

Analogously, passing to the limit as  $k \rightarrow \infty$  in (2.17) we get

$$\alpha \geq \frac{1}{2} \mathcal{I}(u, p) + \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} |p|^2 - \frac{1}{a} (|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi \mathcal{A}^+ \cdot \mathcal{A}^-) \right] dx + \frac{1}{2} \int_{\Omega_0} \mathcal{B} dx.$$

Putting the two above inequalities together we obtain the system of inequalities

$$\begin{aligned} \frac{1}{2} \int_{\Omega_0} \left[ -a |\varepsilon(u)|^2 \right] dx + \int_{\Omega_0} p \cdot \varepsilon(u) dx &\geq \\ &\geq \alpha + \frac{1}{2} \int_{\Omega_0} p \cdot \varepsilon(u) dx - \mathcal{I}(u, p) - \int_{\Omega_0} \mathcal{B} dx \geq \\ &\geq \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} |p|^2 - \frac{1}{a} (|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi \mathcal{A}^+ \cdot \mathcal{A}^-) \right] dx. \end{aligned} \quad (4.7)$$

But from the fact that  $p = a \varepsilon(u) + \mathcal{A}^+ + \psi \mathcal{A}^-$  in  $\Omega_0$  it follows

$$\begin{aligned} \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} |p|^2 - \frac{1}{a} (|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi \mathcal{A}^+ \cdot \mathcal{A}^-) \right] dx &= \\ &= \frac{1}{2} \int_{\Omega_0} \left[ a |\varepsilon(u)|^2 + \frac{1}{a} |\mathcal{A}^+ + \psi \mathcal{A}^-|^2 + 2(\mathcal{A}^+ + \psi \mathcal{A}^-) \cdot \varepsilon(u) + \right. \\ &\quad \left. - \frac{1}{a} (|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi \mathcal{A}^+ \cdot \mathcal{A}^-) \right] dx = \\ &= \frac{1}{2} \int_{\Omega_0} \left[ -a |\varepsilon(u)|^2 + 2(a |\varepsilon(u)|^2 + (\mathcal{A}^+ + \psi \mathcal{A}^-) \cdot \varepsilon(u)) + \frac{1}{a} (\psi^2 - 1) |\mathcal{A}^-|^2 \right] dx = \\ &= \frac{1}{2} \int_{\Omega_0} \left[ -a |\varepsilon(u)|^2 \right] dx + \int_{\Omega_0} p \cdot \varepsilon(u) dx - 2 \int_{\Omega_0} \frac{1}{a} \chi_a \chi_b |\mathcal{A}^-|^2 dx. \end{aligned}$$

Here we used the fact that  $\psi^2 - 1 = -4\chi_a \chi_b$ . Thus in view of (4.7) it follows

$$\begin{aligned} 0 \geq \alpha - \frac{3}{2} \int_{\Omega_0} p \cdot \varepsilon(u) dx + \int_{\Omega_0} a |\varepsilon(u)|^2 dx - \mathcal{I}(u, p) - \int_{\Omega_0} \mathcal{B} dx &\geq \\ &\geq -2 \int_{\Omega_0} \frac{1}{a} \chi_a \chi_b |\mathcal{A}^-|^2 dx. \end{aligned} \quad (4.8)$$

Since

$$\int_{\Omega_0} p \cdot \varepsilon(u) dx = \int_{\Omega_0} \left[ a |\varepsilon(u)|^2 + (\mathcal{A}^+ + \psi \mathcal{A}^-) \cdot \varepsilon(u) \right] dx$$

and in  $\Omega_0$   $\mathcal{A}^- = a \frac{D-C}{2}$  we have

$$0 \geq \alpha - \int_{\Omega_0} [(\mathcal{A}^+ + \psi \mathcal{A}^-) \cdot \varepsilon(u) + \mathcal{B}] dx - \mathcal{I}(u, p) \geq -\frac{1}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx. \quad (4.9)$$

Thus we are allowed to conclude that there exists a  $\theta \in [0, 1]$  such that

$$\begin{aligned} \alpha = & \int_{\Omega_0} [(\mathcal{A}^+ + \psi \mathcal{A}^-) \cdot \varepsilon(u) + \mathcal{B}] dx \\ & + \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u) + \frac{bD-aC}{b-a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\ & \left. - \psi \frac{ab|C-D|^2}{2(b-a)} \right] dx - \frac{\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx. \end{aligned}$$

The obtained result can be summarized as follows.

**Theorem 3.** *Let  $u \in H_0^1(\Omega; \mathbb{R}^n)$  and  $p \in L^2(\Omega; \mathbb{R}_{sym}^{n \times n})$  are the weak limits of  $\{u^k\}$  and  $\{p^k\}$  as defined by (2.8), respectively. Then there exists  $\theta \in [0, 1]$  such that*

$$\begin{aligned} \alpha = & \int_{\Omega_0} \left[ \left( \frac{a(C+D)}{2} + \psi \frac{a(D-C)}{2} \right) \cdot \varepsilon(u) + \frac{a(|C|^2 + |D|^2)}{2} \right. \\ & \left. + \psi \frac{a(|D|^2 - |C|^2)}{2} \right] dx \\ & + \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u) + \frac{bD-aC}{b-a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\ & \left. - \psi \frac{ab|C-D|^2}{2(b-a)} \right] dx - \frac{\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx. \end{aligned} \quad (4.10)$$

Using the fact that in  $\Omega_0$  there holds the equality

$$\int_{\Omega_0} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \varepsilon(u) dx = \int_{\Omega_0} [-a |\varepsilon(u)|^2] dx + \int_{\Omega_0} p \cdot \varepsilon(u) dx,$$



we get from the above formula

$$\begin{aligned}
\alpha = & \int_{\Omega_0} \left[ -a |\varepsilon(u)|^2 + \frac{a(|C|^2 + |D|^2)}{2} + \psi \frac{a(|D|^2 - |C|^2)}{2} \right] dx + \int_{\Omega_0} p \cdot \varepsilon(u) dx + \\
& + \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C - D)}{b - a} \cdot \varepsilon(u) + \frac{bD - aC}{b - a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b - a)} \right. \\
& \left. - \psi \frac{ab|C - D|^2}{2(b - a)} \right] dx - \frac{\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx
\end{aligned} \tag{4.11}$$

and from (4.11) and again the equality  $\int_{\Omega_0} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \varepsilon(u) dx = \int_{\Omega_0} [-a |\varepsilon(u)|^2] dx + \int_{\Omega_0} p \cdot \varepsilon(u) dx$ ,

$$\begin{aligned}
\alpha = & \int_{\Omega_0} \frac{1}{a} |p|^2 dx - \int_{\Omega_0} p \cdot \varepsilon(u) dx + \\
& + \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C - D)}{b - a} \cdot \varepsilon(u) + \frac{bD - aC}{b - a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b - a)} \right. \\
& \left. - \psi \frac{ab|C - D|^2}{2(b - a)} \right] dx + \frac{2-\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx.
\end{aligned} \tag{4.12}$$

By adding (4.11) and (4.12) we get also the formula:

$$\begin{aligned}
\alpha = & \frac{1}{2} \int_{\Omega_0} \left[ \frac{1}{a} |p|^2 - a |\varepsilon(u)|^2 + \frac{a(|C|^2 + |D|^2)}{2} + \psi \frac{a(|D|^2 - |C|^2)}{2} \right] dx + \\
& + \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C - D)}{b - a} \cdot \varepsilon(u) + \frac{bD - aC}{b - a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b - a)} \right. \\
& \left. - \psi \frac{ab|C - D|^2}{2(b - a)} \right] dx + \frac{1-\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx.
\end{aligned} \tag{4.13}$$

Before formulating next theorem it will be convenient to introduce some notation. Denote by  $\omega_0^+$  and  $\omega_0^-$  such subsets of  $\Omega$ , that  $\psi^k \rightarrow 1$  weakly in  $L^1(\omega_0^+)$  and  $\psi^k \rightarrow -1$  weakly in  $L^1(\omega_0^-)$ . Let  $\omega_0 := \omega_0^+ \cup \omega_0^-$ .

**Theorem 4.** *Let  $\nu = \{\nu_x\}_{x \in \Omega}$  be the parametrized Young measure associated to the mi-*

minimization sequence  $\{u^k\}$ . Then

$$\begin{aligned}
\alpha &= \int_{\Omega} \int_{\mathbb{R}^n} h(x, \lambda) d\nu_x(\lambda) dx = \\
&= \int_{\Omega \setminus \Omega_0} \left[ \frac{ab(C-D)}{b-a} \cdot \varepsilon(u) + \frac{bD-aC}{b-a} \cdot p + \frac{ab(|C|^2 - |D|^2)}{2(b-a)} \right. \\
&\quad \left. - \psi \frac{ab|C-D|^2}{2(b-a)} \right] dx \\
&\quad + \int_{\Omega_0} \left[ - \int_{\mathbb{R}^{n \times n}} a|\lambda|^2 d\nu_x(\lambda) + \frac{a(|C|^2 + |D|^2)}{2} + \psi \frac{a(|D|^2 - |C|^2)}{2} \right] dx \\
&\quad + \int_{\Omega_0} p \cdot \varepsilon(u) dx, \tag{4.14}
\end{aligned}$$

where  $h(x, \lambda) = \min \left\{ \frac{1}{2}a(x) |\lambda + C(x)|^2, \frac{1}{2}b(x) |\lambda + D(x)|^2 \right\}$ ,  $\lambda \in \mathbb{R}_{sym}^{n \times n}$ ,  $x \in \Omega$ . Moreover, we have

$$\nu_x = \delta_{\varepsilon(u(x))} \quad a.e. \text{ in } \omega_0. \tag{4.15}$$

*Proof.* By the results expressed in equations (2.15), (2.16), (2.17), (4.4) and lemma 2 we have to compute only the weak limit of the sequence  $\{h_1^k\}$ , where

$$h_1^k = a(x) |\varepsilon(u^k(x))|^2.$$

The sequence  $\{p^k \cdot \varepsilon(u^k)\}$  as being weakly convergent in  $L^1(\Omega)$  has to be equiintegrable according to the Dunford-Pettis criterion of weak compactness in  $L^1(\Omega)$ . Since  $p^k \cdot \varepsilon(u^k) = m^k |\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k)$ , it can be easy to deduce that  $\{m^k |\varepsilon(u^k)|^2\}$  is equiintegrable as well (and so is  $\{h_1^k\}$ ). Thus one can suppose that it is weakly convergent in  $L^1(\Omega)$ , by passing to a subsequence, if necessary, so by Theorem 6.2, p. 97, [Ped97] we see, that its weak limit is  $\int_{\mathbb{R}^{n \times n}} a |\lambda|^2 d\nu_x(\lambda)$ .

Now, from the inequality (R)

$$h(x, \varepsilon(u^k(x))) \leq \frac{1}{2} m^k |\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) + \frac{1}{2} \mathcal{B}^k$$

we are allowed to conclude that the sequence  $\{h(x, \varepsilon(u^k(x)))\}$  has the same property. As shown in (Theorem 6.2, p. 97, [Ped97]) the weak limit is then a function as just given on the right hand side of (4.14).

To show (4.15) it is enough to establish the strong convergence of  $\{\varepsilon(u^k)\}$  in  $L^2(\omega_0; \mathbb{R}_{sym}^{n \times n})$  (cf. Proposition 6.12, p. 111, [Ped97]). The elements of the sequence  $\{\psi^k\} = \{\chi_b^k - \chi_a^k\}$  take values  $+1$  or  $-1$ . Thus the upper Kuratowski limit of the sequence of singletons  $\{\psi^k(x)\}$  (i.e. the set of limit points of this sequence) is the set  $\{-1, 1\}$ . By the Balder theorem (see [Val94]) we see that  $\psi^k \rightarrow 1$  strongly in  $L^1(\omega_0^+)$  and  $\psi^k \rightarrow -1$  strongly in  $L^1(\omega_0^-)$  and we can suppose that  $\psi^k \rightarrow 1$  a.e. in  $\omega_0^+$  ( $\psi^k \rightarrow -1$  a.e. in  $\omega_0^-$ ) by passing to a subsequence, if necessary. Further, the equiintegrability of  $\{m^k |\varepsilon(u^k)|^2\}$  implies that  $\{|\varepsilon(u^k)|^2\}$  is also equiintegrable. By Lemma 2 we have

$$\int_{\omega_0^+} p^k \cdot \varepsilon(u^k) dx \rightarrow \int_{\omega_0^+} p \cdot \varepsilon(u) dx = \int_{\omega_0^+} b |\varepsilon(u)|^2 dx + \int_{\omega_0^+} (\mathcal{A}^+ + \mathcal{A}^-) \cdot \varepsilon(u) dx.$$

On the other hand,

$$\int_{\omega_0^+} p^k \cdot \varepsilon(u^k) dx = \int_{\omega_0^+} b |\varepsilon(u^k)|^2 dx + \int_{\omega_{0k}^-} (a - b) |\varepsilon(u^k)|^2 dx + \int_{\omega_0^+} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) dx,$$

where  $\omega_{0k}^- = \{x \in \omega_0^+ : \psi^k(x) = -1\}$ . Thus taking into account that

$$\int_{\omega_0^+} (\mathcal{A}^+ + \psi^k \mathcal{A}^-) \cdot \varepsilon(u^k) dx \rightarrow \int_{\omega_0^+} (\mathcal{A}^+ + \mathcal{A}^-) \cdot \varepsilon(u) dx$$

and

$$\int_{\omega_{0k}^-} (a - b) |\varepsilon(u^k)|^2 dx \rightarrow 0$$

being a consequence of the equiintegrability of  $\{|\varepsilon(u^k)|^2\}$  and  $|\omega_{0k}^-| \rightarrow 0$ , we are led to

$$\int_{\omega_0^+} b |\varepsilon(u^k)|^2 dx \rightarrow \int_{\omega_0^+} b |\varepsilon(u)|^2 dx.$$

Since, simultaneously,  $\varepsilon(u^k) \rightharpoonup \varepsilon(u)$  in  $L^2(\omega_0^+; \mathbb{R}_{sym}^{n \times n})$ , the desired strong convergence results. Analogous reasoning holds for  $\omega_0^-$ . The proof is complete.  $\square$

**Example.** Let  $\omega_0 = \{x \in \Omega : \chi_a(x)\chi_b(x) = 0\}$ . Without loss of generality one can suppose that  $\psi = 1$  a.e. in  $\omega_0$ . Let  $\omega_{0k}^- = \{x \in \omega_0 : \psi^k(x) = -1\}$ . Since  $\psi^k \rightarrow 1$  weak\* in  $L^\infty(\omega_0)$ , we have

$$2 |\omega_{0k}^-| = \int_{\omega_0} (1 - \psi_k) dx \rightarrow 0. \quad (4.16)$$

Thus  $\psi^k \rightarrow 1$  strongly in  $L^1(\omega_0)$  (in fact, in  $L^p(\omega_0)$  for any  $p \geq 1$ ). By the above theorem this means that  $\nu_x = \delta_{\varepsilon(u(x))}$  a.e. in  $\omega_0$ .

*Remark 5.* From (4.11) and (4.14) it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega_0} a |\varepsilon(u^k)|^2 dx &= \int_{\Omega_0} \int_{\mathbb{R}^n} a |\lambda|^2 d\nu_x(\lambda) dx = \int_{\Omega_0} a |\varepsilon(u)|^2 dx \\ &\quad + \theta \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx, \end{aligned}$$

giving rise to the formula that allows to calculate  $\theta \in [0, 1]$ . Namely, if we let

$$d := \lim_{k \rightarrow \infty} \int_{\Omega_0} a |\varepsilon(u^k)|^2 dx - \int_{\Omega_0} a |\varepsilon(u)|^2 dx, \quad (4.17)$$

then from the equation

$$d = \frac{\theta}{2} \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx$$

we obtain

$$\theta = \begin{cases} \frac{2d}{\int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx} & \text{if } \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (4.18)$$

or equivalently

$$\theta = \begin{cases} \frac{2 \int_{\Omega_0} \int_{\mathbb{R}^n} a |\lambda|^2 d\nu_x(\lambda) dx - \int_{\Omega_0} a |\varepsilon(u)|^2 dx}{\int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx} & \text{if } \int_{\Omega_0} \chi_a \chi_b a |C - D|^2 dx > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (4.19)$$

*Remark 6.* It is worth to point out that the formulas (4.11), (4.12), (4.13) make possible to express the infimum of (P) via (4.18) in terms of the limits  $u$ ,  $p$ ,  $\chi_a$ ,  $\chi_b$ ,  $d$  only. On the other hand, the formula (4.14) expresses it in terms of the parametrized Young measures  $\{\nu_x(\cdot)\}$  which, in practice, are much more difficult to derive.

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